Chapter 2

Taylor Series Reference

A central goal of applied mathematics (even outside of numerics!) is to calculate approximate solutions to problems too hard to find exact solutions, and to quantify the error in these approximations. One methodology for doing this is known as a series expansion. The most straight forward of these is known a Taylor Expansion, in which a function $f(x_0 + \Delta x)$ is represented by a sum of powers of Δx, weighted by the derivatives evaluated at x_0, provided the function is 'nice' enough. The formula in its present form is

$$f(x_0 + \Delta x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (\Delta x)^k \tag{2.1}$$

(Please note that we are taking $0! = 1$.) While this identity is a useful tool, typically we separate the summation into the form

$$\begin{aligned} f(x_0 + \Delta x) &= \textstyle\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (\Delta x)^k + \sum_{k=n}^{\infty} \frac{f^{(k)}(x_0)}{k!} (\Delta x)^k && (2.2) \\ &= \textstyle\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (\Delta x)^k + R_{(n+1)}(x_0, \Delta x) && (2.3) \end{aligned}$$

where we consider $R_{(n+1)}$ to be an error or remainder term in the approximation

$$f(x_0 + \Delta x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!} (\Delta x)^k \tag{2.4}$$

To simplify the error term, we can note that for $\Delta x << 1$,

$$R_{(n+1)}(x_0, \Delta x) = \sum_{k=n}^{\infty} \frac{f^{(k)}(x_0)}{k!} (\Delta x)^k \leq \left| \frac{\max f^{(n+1)}}{(n+1)!} (\Delta x)^{n+1} \right| \tag{2.5}$$

This shouldn't seem too mysterious, since if $\Delta x << 1$, then $\Delta x^{n+1} > \Delta x^{n+2}$. In other words, all higher order terms will contribute less than the Δx^{n+1} term can, at least in the limit as $\Delta x \to 0$.

It is very important to note that this series does not always converge for large Δx, but often works well enough for if we consider small enough Δx. This point is essentially where pure and applied math techniques might diverge slightly. In a pure math course one might go about calculating these derivatives, then finding a range of Δx for which the formulas actually will converge.

An applied mathematician is usually content to assume a function is nice enough to that a small Δx, generated by moving slowly along a function f through approximations such as

$$\begin{aligned} f(x_0) && (2.6) \\ f(x_1) = f(x_0 + \Delta x) && (2.7) \\ f(x_2) = f(x_1 + \Delta) = f((x_0 + \Delta x) + \Delta x) && (2.8) \end{aligned}$$

and so on, will create a good enough approximation to the desired function.

From this sort of setup, these data points(the series of $f(x_n)$ values, say from an experiment) can be used to construct the function's derivatives, or the function's derivatives(say from a differential equation-based theory) can be used to construct the functions themselves.

For this course, most of the subtlety of Taylor series is neglected and the definition is the primary tool we will use to derive other more advanced tools.